* Imp Formulae

1. $$\frac{d}{dx}\left[\frac{1}{F(x)}\right] = \frac{-1}{[F(x)]^2} F'(x)$$

2. $$\frac{d}{dx}\left[\frac{1}{[F(x)]^n}\right] = \frac{-n}{[F(x)]^{n+1}} F'(x)$$

3. $$\int e^{ax} \cdot \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

4. $$\int e^{ax} \cdot \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

5. $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{DN' - ND'}{D^2} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

Unit I: Laplace Transform

Definition - If $F(t)$ is any function then Laplace transform is

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt \quad , \quad s > 0$$

$$= \bar{F}(s).$$

- $F(t) = 1$

$$\therefore L[1] = \int_0^\infty e^{-st} (1)\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}\left[0 - e^0\right]$$

$$= \frac{-1}{s}\left[0 - 1\right]$$

$$= \frac{1}{s}$$

$$\therefore L[1] = \frac{1}{s}$$

Similarly,

$$L[2] = \frac{2}{s}$$

$$L[3] = \frac{3}{s}$$

$$L[k] = \frac{k}{s}$$

Imp

Formulae

1. $L[1] = \frac{1}{s}$ $\rightarrow L[3] = \frac{3}{s}$, $L[k] = \frac{k}{s}$

2. $L[e^{at}] = \frac{1}{s-a}$ $\rightarrow L[e^{2t}] = \frac{1}{s-2}$, $L[e^{t}] = \frac{1}{s-1}$

3. $L[e^{-at}] = \frac{1}{s+a}$ $\rightarrow L[e^{-4t}] = \frac{1}{s+4}$, $L[e^{-t/2}] = \frac{1}{s+1/2}$

4. $L[\sin at] = \frac{a}{s^2+a^2}$ $\rightarrow L[\sin 2t] = \frac{2}{s^2+4}$

5. $L[\cos at] = \frac{s}{s^2+a^2}$ $\rightarrow L[\cos t] = \frac{s}{s^2+1}$

6. $L[\sinh at] = \frac{a}{s^2-a^2}$

7. $L[\cosh at] = \frac{s}{s^2-a^2}$

8. $L[t^n] = \frac{n!}{s^{n+1}}$ $\rightarrow L[t^3] = \frac{3!}{s^4} = \frac{3\times2\times1}{s^4} = \frac{6}{s^4}$

$= \frac{\Gamma(n+1)}{s^{n+1}}$ $\rightarrow L[t^{1/2}] = \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{\Gamma(3/2)}{s^{3/2}}$

Now, $\Gamma(3/2) = 1/2\,\Gamma(1/2) = 1/2\sqrt{\pi}$

$\boxed{\text{Note :- } \Gamma(1/2) = \sqrt{\pi}}$

$$\Gamma\frac{5}{2} = \frac{3}{2} \times \Gamma\frac{3}{2}$$

$$= \frac{3}{2} \times \frac{1}{2} \times \Gamma\frac{1}{2}$$

$$= \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$$

$$\Gamma\frac{5}{2} = \frac{3\sqrt{\pi}}{4}$$

* Formulae.

1. $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

2. $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

3. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$

4. $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

5. $\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$$= \frac{2\tan\theta}{1+\tan^2\theta}$$

6. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$\therefore 2\sin^2\theta = 1 - \cos 2\theta$

$$\boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$\therefore \quad 2\cos^2\theta = 1 + \cos 2\theta$

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore \quad 4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$$\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$4\cos^3\theta = \cos 3\theta + 3\cos\theta$

$$\boxed{\cos^3\theta = \frac{1}{4}\left[\cos 3\theta + 3\cos\theta\right]}$$

Examples -

1. $L[\sin^2 2t]$

We know,

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

θ replace by $2t$

$$\sin^2 2t = \frac{1-\cos 2\times 2t}{2}$$

$$\sin^2 2t = \frac{1-\cos 4t}{2}$$

$$\sin^2 2t = \frac{1}{2}[1-\cos 4t]$$

$$L[\sin^2 2t] = \frac{1}{2}L[1-\cos 4t]$$

$$\boxed{L[\sin^2 2t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]}$$

H.W. 1) $a = 3t$, $a = \frac{1}{2}t$

$a = 3t$

1. $L[\sin^2 3t]$

We know.

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

θ replace by $3t$

$$\sin^2 3t = \frac{1-\cos 2\times 3t}{2}$$

$$\sin^2 3t = \frac{1}{2}[1-\cos 6t]$$

$$L[\sin^2 3t] = \frac{1}{2}L[1-\cos 6t]$$

$$\boxed{L[\sin^2 3t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]}$$

2. $L[\cos^2 3t]$

We know,

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1}{2}[1+\cos 2\theta]$$

replace θ by $3t$

$$\cos^2 3t = \frac{1}{2}[1+\cos 2\times 3t]$$

$$\cos^2 3t = \frac{1}{2}[1+\cos 6t]$$

$$L[\cos^2 3t] = \frac{1}{2}L[1+\cos 6t]$$

$$\boxed{L[\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]}$$

3. $L[\sin^3 3t]$

We know,

$$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$$

θ replace by $3t$

$$\sin^3 3t = \frac{1}{4}\left[3\sin 3t - \sin 3\times 3t\right]$$

$$\sin^3 3t = \frac{1}{4}[3\sin 3t - \sin 9t]$$

$$L[\sin^3 3t] = \frac{1}{4} L[3\sin 3t - \sin 9t]$$

$$\therefore L[\sin^3 3t] = \frac{1}{4}\left[3 \times \frac{3}{s^2+9} - \frac{9}{s^2+81}\right]$$

$$= \frac{1}{4}\left[\frac{9}{s^2+9} - \frac{9}{s^2+81}\right]$$

$$\boxed{L[\sin^3 3t] = \frac{9}{4}\left[\frac{1}{s^2+9} - \frac{91}{s^2+81}\right]}$$

4. $L[\cos^3 3t]$

We know,

$$\cos^3\theta = \frac{1}{4}[\cos 3\theta + 3\cos\theta]$$

θ replace by $3t$

$$\cos^3 3t = \frac{1}{4}[\cos 3 \times 3t + 3\cos 3t]$$

$$= \frac{1}{4}[\cos 9t + 3\cos 3t]$$

$$L[\cos^3 3t] = \frac{1}{4} L[\cos 9t + 3\cos 3t]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+81} + 3 \times \frac{s}{s^2+9}\right]$$

$$\boxed{L[\cos^3 3t] = \frac{1}{4}\left[\frac{s}{s^2+81} + \frac{3s}{s^2+9}\right]}$$

$a = \frac{1}{2}t.$

1. $L[\sin^2 \frac{1}{2}t]$

we know,

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

θ replace by $\frac{1}{2}t$

$$\sin^2 \tfrac{1}{2}t = \frac{1-\cos 2\times \frac{1}{2}t}{2}$$

$$\sin^2 \tfrac{1}{2}t = \frac{1-\cos t}{2}$$

$$\sin^2 \tfrac{1}{2}t = \frac{1}{2}[1-\cos t]$$

$$L[\sin^2 \tfrac{1}{2}t] = \frac{1}{2} L[1-\cos t]$$

$$\boxed{L[\sin^2 \tfrac{1}{2}t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]}$$

2. $L[\cos^2 \frac{1}{2}t]$

we know,

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

θ replace by $\frac{1}{2}t$

$$\cos^2 \tfrac{1}{2}t = \frac{1+\cos 2\times \frac{1}{2}t}{2}$$

$$\cos^2 \tfrac{1}{2}t = \frac{1 + \cos t}{2}$$

$$\cos^2 \tfrac{1}{2}t = \frac{1}{2}[1 + \cos t]$$

$$L[\cos^2 \tfrac{1}{2}t] = \frac{1}{2} L[1 + \cos t]$$

$$\boxed{L[\cos^2 \tfrac{1}{2}t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+1}\right]}$$

3. $L[\sin^3 \tfrac{1}{2}t]$

We know,

$$\sin^3\theta = \frac{1}{4}[3\sin\theta - \sin 3\theta]$$

θ replace by $\tfrac{1}{2}t$

$$\sin^3 \tfrac{1}{2}t = \frac{1}{4}[3\sin \tfrac{1}{2}t - \sin 3 \times \tfrac{1}{2}t]$$

$$L[\sin^3 \tfrac{1}{2}t] = \frac{1}{4}\left[3 \times \frac{1/2}{s^2 + 1/4} - \frac{3/2}{s^2 + 9/4}\right]$$

$$= \frac{1}{4}\left[\frac{3/2}{s^2 + 1/4} - \frac{3/2}{s^2 + 9/4}\right]$$

$$\boxed{L[\sin^3 \tfrac{1}{2}t] = \frac{3}{8}\left[\frac{1}{s^2 + 1/4} - \frac{1}{s^2 + 9/4}\right]}$$

4. $L[\cos^3 \tfrac{1}{2}t]$

We know,

$$\cos^3\theta = \frac{1}{4}[\cos 3\theta + 3\cos\theta]$$

θ replace by $\tfrac{1}{2}t$

$$\cos^3 \tfrac{1}{2}t = \frac{1}{4}\left[\cos 3\times\tfrac{1}{2}t + 3\cos\tfrac{1}{2}t\right]$$

$$\cos^3 \tfrac{1}{2}t = \frac{1}{4}\left[\cos \tfrac{3}{2}t + 3\cos\tfrac{1}{2}t\right]$$

$$L[\cos^3 \tfrac{1}{2}t] = \frac{1}{4}L\left[\cos \tfrac{3}{2}t + 3\cos\tfrac{1}{2}t\right]$$

$$L[\cos^3 \tfrac{1}{2}t] = \frac{1}{4}\left[\frac{s}{s^2+9/4} + 3\times\frac{s}{s^2+1/4}\right]$$

$$\boxed{L[\cos^3 \tfrac{1}{2}t] = \frac{1}{4}\left[\frac{s}{s^2+9/4} + \frac{3s}{s^2+1/4}\right]}$$

Imp

Formulae -

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\cdot\sin B = \sin(A+B) - \sin(A-B)$

$2\sin A\cdot\sin B = \cos(A-B) - \cos(A+B)$

Example.

1. $L[\cos t\cos 2t] =$

we know.

$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$2\cos t \;\; \cos 2t = \cos(t+2t) + \cos(t-2t)$

$= \cos 3t + \cos(-t)$

$2\cos t \cdot \cos 2t = \cos 3t + \cos t$

$\cos t \cdot \cos 2t = \frac{1}{2}[\cos 3t + \cos t]$

Taking L.T on both sides,

$L[\cos t \cdot \cos 2t] = \frac{1}{2} L[\cos 3t + \cos t]$

$$\boxed{L[\cos t \cdot \cos 2t] = \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right]}$$

H.W. $L[\cos t \cdot \cos 2t \cdot \cos 3t]$

1. $L[\cos t \; \cos 2t \cdot \cos 3t]$

We know

$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$2\cos t \cdot \cos 2t = \cos(t+2t) + \cos(t-2t)$

$= \cos 3t + \cos(-t)$

$2\cos t \cdot \cos 2t = \cos 3t + \cos t$

$\cos t \cdot \cos 2t = \frac{1}{2}[\cos 3t + \cos t]$

$\cos t \cdot \cos 2t \cdot \cos 3t = \frac{1}{2}[\cos 3t + \cos t] \cos 3t$

$\cos t \cdot \cos 2t \cdot \cos 3t = \frac{1}{2}[\cos^2 3t + \cos t \cdot \cos 3t]$ — 1

consider

$\cos^2 3t = \frac{1 + \cos 2\times 3t}{2}$

$= \frac{1}{2}[1 + \cos 6t]$ — 2

consider

cost cos3t

we know,

$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$2\cos t \cdot \cos 3t = \cos(t+3t) + \cos(t-3t)$

$= \cos 4t + \cos(-2t)$

$2\cos t \cdot \cos 3t = \cos 4t + \cos 2t$

$\cos t \cdot \cos 3t = \frac{1}{2}[\cos 4t + \cos 2t]$ — ③

Put eqⁿ ② & ③ in eqⁿ ①

$\cos t \cos 2t \cdot \cos 3t = \frac{1}{2}\left\{\left[\frac{1}{2}(1+\cos 6t)\right] + \left[\frac{1}{2}(\cos 4t + \cos 2t)\right]\right\}$

$= \frac{1}{4}(1+\cos 6t) + \frac{1}{4}(\cos 4t + \cos 2t)$

$\cos t \cdot \cos 2t \cos 3t = \frac{1}{4}\left[1 + \cos 6t + \cos 4t + \cos 2t\right]$

Taking L.T on both sides

$L[\cos t \cos 2t \cdot \cos 3t] = \frac{1}{4} L\left[1 + \cos 6t + \cos 4t + \cos 2t\right]$

$= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$

$\therefore L[\cos t \cdot \cos 2t \cdot \cos 3t] = \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$

2. $L[\sin 2t \cdot \cos 3t]$

→

$2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$

$2\sin 2t \cdot \cos 3t = \sin(2t+3t) + \sin(2t-3t)$

$2\sin 2t \cdot \cos 3t = \sin(5t) + \sin(-t)$

$\sin 2t \cdot \cos 3t = \frac{1}{2}[\sin 5t - \sin t]$

Taking L.T ...

$$\boxed{L[\sin 2t \cdot \cos 3t] = \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]}$$

3. $L[\sinh^3 2t]$

We know,

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh 2t = \frac{e^{2t} - e^{-2t}}{2}$$

$$\sinh^3 2t = \left[\frac{e^{2t} - e^{-2t}}{2}\right]^3$$

$$(a \mp b)^3 = a^3 \mp b^3 \mp 3a^2b \pm 3ab^2$$

$$\sinh^3 2t = \frac{1}{8}\left[(e^{2t})^3 - (e^{-2t})^3 - 3(e^{2t})^2(e^{-2t}) + 3e^{2t}(e^{-2t})^2\right]$$

$$= \frac{1}{8}\left[e^{6t} - e^{-6t} - 3e^{4t}e^{-2t} + 3e^{2t}e^{-4t}\right]$$

$$= \frac{1}{8}\left[e^{6t} - e^{-6t} - 3e^{2t} + 3e^{-2t}\right]$$

taking L.T on both sides

$$L[\sinh^3 2t] = \frac{1}{8} L\left[e^{6t} - e^{-6t} - 3e^{2t} + 3e^{-2t}\right]$$

$$\boxed{L[\sinh^3 2t] = \frac{1}{8}\left[\frac{1}{s-6} - \frac{1}{s+6} - \frac{3}{s-2} + \frac{3}{s+2}\right]}$$

First Shift Theorem -

If $L[F(t)] = \bar{F}(s)$

$L[e^{at} F(t)] = \bar{F}(s-a)$

$L[e^{-at} F(t)] = \bar{F}(s+a)$

1. $L[e^{2t} \sin 3t]$

Let $F(t) = \sin 3t$

$L[F(t)] = L[\sin 3t] = \frac{3}{s^2+9} = \bar{F}(s)$

$L[e^{at} F(t)] = \bar{F}(s-a)$

$$\boxed{L[e^{2t} \sin 3t] = \frac{3}{(s-2)^2+9}}$$

2. $L[e^{-3t} (\cos 4t + 3\sin 4t)]$

Let $F(t) = \cos 4t + 3\sin 4t$

$L[F(t)] = L[\cos 4t + 3\sin 4t]$

$= \frac{s}{s^2+16} + \frac{3 \times 4}{s^2+16}$

$= \frac{12+s}{s^2+16}$

$= \bar{F}(s)$

By first Shift Theorem,

$L[e^{-3t} (\cos 4t + 3\sin 4t)] = \bar{F}(s+3)$

$= \frac{12+s+3}{(s+3)^2+16}$

$= \frac{15+s}{(s+3)^2+16}$

H.W.

1. $L[e^{-t} \sin^2 2t]$

Let $F(t) = \sin^2 2t$

$$L[F(t)] = \frac{1}{2} L[1 \mp \cos 4t]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$= \bar{F}(s)$$

By first shift theorem.

$$L[e^{-t} \sin^2 2t] = \bar{F}(s+1)$$

$$\therefore \boxed{L[e^{-t} \sin^2 2t] = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+16}\right]}$$

2 $L[\sin^4 t\, e^{2t}]$

Let $F(t) = \sin^4 t$

$$= (\sin^2 t)^2$$

$$= \left(\frac{1-\cos 2t}{2}\right)^2$$

$$= \frac{1}{4}(1-\cos 2t)^2$$

Use $(a-b)^2 = a^2 - 2ab + b^2$

$$= \frac{1}{4}\left[1 - 2\cos 2t + \cos^2 2t\right]$$

$$= \frac{1}{4}\left[1 - 2\cos 2t + \frac{1}{2}(1+\cos 4t)\right]$$

$$= \frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{8} + \frac{1}{8}\cos 4t$$

Taking L.T on both sides.

$$L[\sin^4 t] = L\left[\frac{1}{4} - \frac{1}{2}\cos 2t + \frac{1}{8} + \frac{1}{8}\cos 4t\right]$$

$$= \frac{1}{4s} - \frac{1}{2} \times \frac{s}{s^2+4} + \frac{1}{8s} + \frac{1}{8} \times \frac{s}{s^2+16}$$

$$L[\sin^4 t] = \frac{1}{4s} - \frac{s}{2(s^2+4)} + \frac{1}{8s} + \frac{s}{8(s^2+16)}$$

$$= \frac{1}{4}\left[\frac{1}{s} - \frac{2s}{s^2+4} + \frac{1}{2s} + \frac{s}{2(s^2+16)}\right]$$

$$= \frac{1}{4}\left[\frac{3}{2s} - \frac{2s}{s^2+4} + \frac{s}{2(s^2+16)}\right]$$

$$= \frac{3}{8s} - \frac{s}{2(s^2+4)} + \frac{s}{8(s^2+16)}$$

$$= \bar{F}(s)$$

$$\therefore L[e^{2t}\sin^4 t] = \frac{3}{8(s-2)} - \frac{s-2}{2[(s-2)^2+4]} + \frac{s-2}{8[(s-2)^2+16]}$$

H.W.

$L[e^{-2t}\sin^3 t]$

→ Let $F(t) = \sin^3 t$

$$\sin^3 t = \frac{1}{4}[3\sin t - \sin 3t]$$

Taking L.T on both sides

$$L[\sin^3 t] = \frac{1}{4} L[3\sin t - \sin 3t]$$

$$= \frac{1}{4}\left[3 \times \frac{1}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$= \frac{1}{4}\left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$= \bar{F}(s)$$

$$\therefore \boxed{L[e^{-2t}\sin^3 t] = \frac{1}{4}\left[\frac{3}{(s+2)^2+1} - \frac{3}{(s+2)^2+9}\right]}$$

Division by 't'.

If $L[F(t)] = \bar{F}(s)$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}(s)\,ds$$

formulae -

1) $\int \frac{F'(x)}{F(x)}\,dx = \log F(x)$

2) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

3) $\tan^{-1}x + \cot^{-1}x = \pi/2$

4) $\tan^{-1}(\infty) = \pi/2$

5) $\log a - \log b = \log(a/b)$, $a\log b = \log b^a$

1. $L\left[\frac{\sin 2t}{t}\right]$

$\rightarrow$ Let $f(t) = \sin 2t$

$$L[f(t)] = L[\sin 2t]$$

$$= \frac{2}{s^2+4}$$

$$= \bar{F}(s)$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{F}(s)\, ds$$

$$L\left[\frac{\sin 2t}{t}\right] = \int_s^\infty \frac{2}{s^2+2^2}\, ds$$

$$= 2\int_s^\infty \frac{1}{s^2+2^2}\, ds$$ - use $\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$= 2 \times \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/2)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$ $\because \tan^{-1}(\infty) = \pi/2$

Use $\tan^{-1} x + \cot^{-1} x = \pi/2$

$\therefore \frac{\pi}{2} - \tan^{-1} x = \cot^{-1} x$

$$\boxed{L\left[\frac{\sin 2t}{t}\right] = \cot^{-1}\left(\frac{s}{2}\right)}$$

2) $L\left[\frac{\cos 2t}{t}\right]$

→ Let $F(t) = \cos 2t$

$$L[F(t)] = L[\cos 2t]$$

$$= \frac{s}{s^2+4}$$

$$= \bar{F}(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}(s)\, ds$$

$$L\left[\frac{\cos 2t}{t}\right] = \int_s^\infty \frac{s}{s^2+2^2}\, ds$$

2s

$$= \int_s^\infty \frac{s}{s^2+4}\, ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}\, ds$$

$$\int \frac{F'(x)}{F(x)}\, dx = \log F(x)$$

$$= \frac{1}{2}\left[\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log(s^2+4)\right]$$

$$= \frac{-1}{2}\log(s^2+4)$$

$$a \log b = \log b^a$$

$$= \log (s^2+4)^{-1/2}$$

$$= \log \left(\frac{1}{(s^2+4)^{1/2}} \right)$$

$$\boxed{L\left[\frac{\cos 2t}{t}\right] = \log \left(\frac{1}{\sqrt{s^2+4}} \right)}$$

* H.W.

3. $L\left[\frac{\sin 2t . \cos t}{t}\right]$

→ Let $F(t) = \sin 2t \cdot \cos t$.

we know,

$2 \sin A \cdot \cos B = \sin (A+B) + \sin (A-B)$

$\therefore 2 \sin 2t \cdot \cos t = \sin (2t+t) + \sin (2t-t)$

$2 \sin 2t \cdot \cos t = \sin 3t + \sin t$

$\sin 2t \cdot \cos t = \frac{1}{2} [\sin 3t + \sin t]$

Taking L.T on both sides

$$L[\sin 2t \cdot \cos t] = \frac{1}{2} L [\sin 3t + \sin t]$$

$$L[\sin 2t \cdot \cos t] = \frac{1}{2} \left[\frac{3}{s^2+9} + \frac{1}{s^2+1} \right]$$

$$= \bar{F}(s)$$

Now,

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}(s)\, ds$$

$$= \int_s^\infty \frac{1}{2} \left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1^2} \right] ds$$

$$= \frac{1}{2} \int_s^{\infty} \left[\frac{3}{s^2+3^2} + \frac{1}{s^2+1^2} \right] ds$$

$$= \frac{1}{2} \left\{ \int_s^{\infty} \frac{3}{s^2+3^2} ds + \int_s^{\infty} \frac{1}{s^2+1^2} ds \right\}$$

$$= \frac{1}{2} \left[3 \int_s^{\infty} \frac{1}{s^2+\underline{3}^2} ds + \int_s^{\infty} \frac{1}{s^2+1^2} ds \right]$$

Use $\int \frac{1}{x^2+a^2} ds = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

$$= \frac{1}{2} \left[3 \left(\frac{1}{3} \tan^{-1} \frac{s}{3} \right)_s^{\infty} + \left(\frac{1}{1} \tan^{-1}\left(\frac{s}{1}\right) \right)_s^{\infty} \right]$$

$$= \frac{1}{2} \left[\left(\tan^{-1}\left(\frac{s}{3}\right) \right)_s^{\infty} + \left(\tan^{-1}(s) \right)_s^{\infty} \right]$$

$$= \frac{1}{2} \left[\left(\tan^{-1}\infty - \tan^{-1}\left(\frac{s}{3}\right) \right) + \left(\tan^{-1}(\infty) - \tan^{-1}(s) \right) \right]$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) \right) + \left(\frac{\pi}{2} - \tan^{-1}(s) \right) \right]$$

Use $\tan^{-1} x + \cot^{-1} x = \pi/2$ $\therefore \frac{\pi}{2} - \tan^{-1} x = \cot^{-1} x$

$$= \frac{1}{2} \left[\cot^{-1}\left(\frac{s}{3}\right) + \cot^{-1}(s) \right]$$

$$\therefore \boxed{L\left[\frac{\sin 2t \cdot \cos t}{t} \right] = \frac{1}{2} \left[\cot^{-1}\left(\frac{s}{3}\right) + \cot^{-1}(s) \right]}$$

4. $L\left[\frac{e^{at} - \cos bt}{t}\right]$

$\rightarrow$ Let $F(t) = e^{at} - \cos bt$

$\therefore L[f(t)] = L[e^{at} - \cos bt]$

$L[F(t)] = \frac{1}{s-a} - \frac{s}{s^2+b^2}$

$= \bar{F}(s)$

Now,

$$L\left[\frac{F(t)}{t}\right] = \int_s^{\infty}\left[\frac{1}{s-a} - \frac{s}{s^2+b^2}\right] ds$$

Use $\int \frac{F'(x)}{F(x)} ds = \log F(x)$

$$= \int_s^{\infty} \frac{1}{s-a} ds - \frac{1}{2}\int_s^{\infty} \frac{2s}{s^2+b^2} ds$$

$$= [\log(s-a)]_s^{\infty} - \frac{1}{2}[\log(s^2+b^2)]_s^{\infty}$$

$$= [\log(\infty) - \log(s-a)] - \frac{1}{2}[\log\infty - \log(s^2+b^2)]$$

$$= 0 - \log(s-a) - \frac{1}{2}[0 - \log(s^2+b^2)]$$

$$= -\log(s-a) + \frac{1}{2}\log(s^2+b^2)$$

$a\log b = \log b^a$

$$= -\log(s-a) + \log(s^2+b^2)^{1/2}$$

$\log a - \log b = \log(a/b)$

$$\boxed{L\left[\frac{e^{at} - \cos bt}{t}\right] = \log\left[\frac{\sqrt{s^2+b^2}}{s-a}\right]}$$

5. $L\left[\dfrac{e^{-at}-e^{-bt}}{t}\right]$

$\rightarrow$ Let $F(t) = e^{-at} - e^{-bt}$

$$L[F(t)] = L[e^{-at} - e^{-bt}]$$
$$= \frac{1}{s+a} - \frac{1}{s+b}$$
$$= \bar{F}(s)$$
$$L\left[\frac{F(t)}{t}\right] = \int_s^{\infty} \bar{F}(s)\, ds$$
$$= \int_s^{\infty} \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

Use $\int \frac{F'(x)}{F(x)} dx = \log(F(x))$

$$= \left[\log(s+a)\right]_s^{\infty} - \left[\log(s+b)\right]_s^{\infty}$$
$$= (0 - \log(s+a)) - (0 - \log(s+b))$$
$$= -\log(s+a) + \log(s+b)$$

$\log a - \log b = \log(a/b)$

$$\boxed{L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left(\frac{s+b}{s+a}\right)}$$

H-W

6. $L\left[\dfrac{e^{t} - \cos t}{t}\right]$

7. $L\left[\dfrac{\sin^2 t}{t}\right]$

Let $F(t) = \sin^2 t$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\therefore \sin^2 t = \frac{1-\cos 2t}{2}$$

$$L[\sin^2 t] = \frac{1}{2} L[1-\cos 2t]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \bar{F}(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}(s)\, ds$$

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds.$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{s}\, ds - \frac{1}{2}\int_s^\infty \frac{s}{s^2+4}\, ds$$

$$= \frac{1}{2}[\log s]_s^\infty - \frac{1}{2} \times \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}\, ds$$

$$= \frac{1}{2}[0 - \log s] - \frac{1}{2} \times \frac{1}{2}\left[\log(s^2+4)\right]_s^\infty$$

$$= -\frac{1}{2}\log s - \frac{1}{2} \times \frac{1}{2}\left[0 - \log(s^2+4)\right]$$

$$= \frac{1}{2}\left[-\log s + \frac{1}{2}\log(s^2+4)\right]$$

$$a \log b = \log b^a$$

$$= \frac{1}{2}\left[-\log s + \log(s^2+4)^{1/2}\right]$$

$$\log a - \log b = \log(a/b)$$

$$= \frac{1}{2}\log\left[\frac{(s^2+4)^{1/2}}{s}\right]$$

$$L\left[\frac{e^{-2t}\sin 3t}{t}\right]$$

Let $F(t) = \sin 3t$

$$L[\sin 3t] = \frac{3}{s^2+9} = \bar{F}_1(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^{\infty} \bar{F}_1(s)\,ds$$

$$L\left[\frac{\sin 3t}{t}\right] = 3\int_s^{\infty} \frac{1}{s^2+3^2}\,ds$$

$$= 3 \times \frac{1}{3}\left(\tan^{-1}\left(\frac{s}{3}\right)\right)_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}(s/3)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$\therefore L\left[\frac{\sin 3t}{t}\right] = \cot^{-1}\left(\frac{s}{3}\right) = \bar{F}_2(s)$$

$$\therefore L\left[\frac{e^{-2t}\sin 3t}{t}\right] = \bar{F}_2(s+2)$$

$$\boxed{L\left[\frac{e^{-2t}\sin 3t}{t}\right] = \cot^{-1}\left(\frac{s+2}{3}\right)}$$

H.W $L\left[\dfrac{e^t - \cos t}{t}\right]$

→ Let $f(t) = e^t - \cos t$

$L[F(t)] = L[e^t - \cos t]$

$L[F(t)] = \dfrac{1}{s-1} - \dfrac{s}{s^2+1}$

$= \bar{F}(s)$

Now,

$$L\left[\frac{e\,f(t)}{t}\right] = \int_s^\infty \bar{F}(s)\,ds$$

$$L\left[\frac{e^t - \cos t}{t}\right] = \int_s^\infty \left[\frac{1}{s-1} - \frac{s}{s^2+1}\right] ds$$

Use $\displaystyle\int \frac{F'(x)}{F(x)}\,dx = \log x$

$$= \int_s^\infty \frac{1}{s-\cancel{a}1}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+1}\,ds$$

$$= \left[\log(s-\cancel{a}1)\right]_s^\infty - \frac{1}{2}\left[\log(s^2+1)\right]_s^\infty$$

$$= [\log\infty - \log(s-1)] - \frac{1}{2}[\log(\infty) - \log(s^2+1)]$$

$$= [0 - \log(s-1)] - \frac{1}{2}[0 - \log(s^2+1)]$$

$$= -\log(s-1) + \frac{1}{2}\log(s^2+1)$$

$$= -\log(s-1) + \log(s^2+1)^{1/2}$$

$$\boxed{L\left[\frac{e^t - \cos t}{t}\right] = \log\left[\frac{(s^2+1)^{1/2}}{s-1}\right]}$$

$$L\left[\frac{e^{-t}\cos 2t}{t}\right]$$

Let $F(t) = \cos 2t$

$$L[\cos 2t] = \frac{s}{s^2+2^2} = \frac{s}{s^2+4} = \bar{F}_1(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}_1(s)\, ds$$

$$= \int_s^\infty \frac{s}{s^2+4}\, ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}\, ds$$

Use $\int \frac{F'(x)}{F(x)}\, dx = \log(F(x))$

$$= \frac{1}{2}\left[\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log(s^2+4)\right]$$

$$= -\frac{1}{2}\log(s^2+4)$$

$$L\left[\frac{\cos 2t}{t}\right] = \log(s^2+4)^{-1/2}$$

$$= \bar{F}_2(s)$$

By first shift theorem.

$$L\left[\frac{e^{-t}\cos 2t}{t}\right] = \bar{F_2}(s+1)$$

$$\boxed{L\left[\frac{e^{-t}\cos 2t}{t}\right] = \log\left[(s+1)^2+4\right]^{-1/2}}$$

Multiply by t^n

If $L[F(t)] = \bar{F}(s)$

$$L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{F}(s)]$$

Formula :-

1) $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{DN' - ND'}{D^2}$

2) $\frac{d}{dx}\left[\frac{1}{F(x)}\right] = \frac{-1}{[F(x)]^2} \times F'(x)$

3) $\frac{d}{dx}\left[\frac{1}{[F(x)]^n}\right] = \frac{-n}{[F(x)]^{n+1}} \times F'(x)$

1. $L[t \cosh 3t]$

$\rightarrow$ Let $F(t) = \cosh 3t$

$$L[F(t)] = L[\cosh 3t]$$

$$= \frac{s}{s^2-9}$$

$$= \bar{F}(s)$$

$$\therefore L[tF(t)] = (-1)\frac{d}{ds}[\bar{F}(s)]$$

$$L[t \cdot \cosh 3t] = (-1)\frac{d}{ds}\left[\frac{s}{s^2-9}\right]$$

Use $\frac{d}{ds}\left(\frac{u}{v}\right) = \frac{DN' - ND'}{D^2}$

$$= (-1)\left[\frac{(s^2-9)\times 1 - s(2s-0)}{(s^2-9)^2}\right]$$

$$= (-1)\left[\frac{s^2-9-2s^2}{(s^2-9)^2}\right]$$

$$= (-1)\left[\frac{-s^2-9}{(s^2-9)^2}\right]$$

$$= -1\left[\frac{-(s^2+9)}{(s^2-9)^2}\right]$$

$$\boxed{L[t \cdot \cosh 3t] = \frac{s^2+9}{(s^2-9)^2}}$$

2. $L[t \sinh 3t]$

Let $F(t) = \sinh 3t$

$$L[F(t)] = L[\sinh 3t]$$

$$= \frac{3}{s^2-9} = \bar{F}(s)$$

$$L[t \cdot F(t)] = (-1)\frac{d}{ds}[\bar{F}(s)]$$

$$= (-1)\frac{d}{ds}\left[\frac{3}{s^2-9}\right]$$

$$= (-1)^1 \times 3 \times \frac{d}{ds}\left[\frac{1}{s^2-9}\right]$$

Use $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-1}{[f(x)]^2} f'(x)$

$$= -3\left[\frac{(-1)}{(s^2-9)^2} \times 2s\right]$$

$$\boxed{L[t.\sinh 3t] = \frac{6s}{(s^2-9)^2}}$$

3) $L[t\sin 3t\cos 2t]$

Let $f(t) = \sin 3t \cdot \cos 2t$

$2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$

$2\sin 3t \cdot \cos 2t = \sin(3t+2t) + \sin(3t-2t)$

$= \sin 5t + \sin t$

$\sin 3t \cdot \cos 2t = \frac{1}{2}[\sin 5t + \sin t]$

$L[\sin 3t \cdot \cos 2t] = \frac{1}{2} L[\sin 5t + \sin t]$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$= \bar{f}(s)$$

$$L[t\sin 3t \cdot \cos 2t] = \frac{1}{2}(-1)\frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$= \frac{-1}{2}\left[\frac{d}{ds}\left(\frac{5}{s^2+25}\right) + \frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{-5}{(s^2+25)^2} \times 2s + \frac{-1}{(s^2+1)^2} \times 2s\right]$$

$$= -\frac{1}{2}\left[\frac{-10s}{(s^2+25)^2} - \frac{2s}{(s^2+1)^2}\right]$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

$$\therefore L[t\sin 3t \cdot \cos 2t] = \frac{5}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

34. $L[te^{-t}\cosh t]$

Let $F(t) = \cosh t$

$$L[F(t)] = L[\cosh t]$$

$$= \frac{s}{s^2-1}$$

$$= \bar{F}_1(s)$$

$$L[tF(t)] = -\frac{d}{ds}[\bar{F}(s)]$$

$$= -\frac{d}{ds}\left[\frac{s}{s^2-1}\right]$$

Use $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{DN' - ND'}{D^2}$

$$= -\left[\frac{(s^2-1)\times 1 - s(2s-0)}{(s^2-1)^2}\right]$$

$$= -\left[\frac{s^2-1-2s^2}{(s^2-1)^2}\right]$$

$$= -\left[\frac{-s^2-1}{(s^2-1)^2}\right]$$

$$= \frac{s^2+1}{(s^2-1)^2}$$

$$= \overline{F_2}(s)$$

$$L[e^{-at}F(t)] = \overline{F_2}(s+a)$$

$$L[e^{-t}t\cosh t] = \overline{F_2}(s+1)$$

$$L[e^{-t}t\cosh t] = \frac{(s+1)^2+1}{((s+1)^2-1)^2}$$

$$\boxed{L[e^{-t}t\cosh t] = \frac{(s+1)^2+1}{[(s+1)^2-1]^2}}$$

5) $L[t.(3\sin 2t - 2\cos 2t)]$

$\longrightarrow$

Let $F(t) = 3\sin 2t - 2\cos 2t$

$$L[F(t)] = L[3\sin 2t - 2\cos 2t]$$

$$= 3 \times \frac{2}{s^2+4} - 2 \times \frac{s}{s^2+4}$$

$$L[3\sin 2t - 2\cos 2t) = \frac{6}{s^2+4} - \frac{2s}{s^2+4} = \overline{F_1}(s)$$

$$L[tF(t)] = -\frac{d}{ds}[\bar{F}_1(s)]$$

$$L[t(3\sin 2t = 2\cos 2t] = -\frac{d}{ds}\left(\frac{6}{s^2+4} - \frac{2s}{s^2+4}\right)$$

$$= -\left[\frac{d}{ds}\left(\frac{6}{s^2+4}\right) - \frac{d}{ds}\left(\frac{2s}{s^2+4}\right)\right]$$

Use $\frac{d}{dx}\left(\frac{1}{F(x)}\right) = -\frac{1}{[F(x)]^2}$,

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{DN' - ND'}{D^2}$

$$= -\left[\left(6 \times \frac{-1}{(s^2+4)^2} \times 2s\right) - \left(\frac{(s^2+4)\times 2 - 2s(2s+0)}{(s^2+4)^2}\right)\right]$$

$$= -\left[\frac{-12s}{(s^2+4)^2} - \left(\frac{2(s^2+4) - 4s^2}{(s^2+4)^2}\right)\right]$$

$$= -\left[\frac{-12s}{(s^2+4)^2} - \left(\frac{2s^2+8-4s^2}{(s^2+4)^2}\right)\right]$$

$$= \frac{12s}{(s^2+4)^2} + \frac{8-2s^2}{(s^2+4)^2}$$

$$- \frac{12s+8-2s^2}{(s^2+4)^2}$$

$$\therefore \boxed{L[t(3\sin 2t - 2\cos 2t)] - \frac{12s+8-2s^2}{(s^2+4)^2}}$$

6) $L[t^3 e^{-3t}]$

$\longrightarrow$ Let $F(t) = e^{-3t}$

$$L[F(t)] = L[e^{-3t}]$$
$$= \frac{1}{s+3}$$
$$= \bar{F}(s)$$

$$L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{F}(s)]$$

$$L[t^3 e^{-3t}] = (-1)^3 \frac{d^3}{ds^3}\left[\frac{1}{s+3}\right]$$

$$= (-1) \frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{1}{s+3}\right)\right]$$

use $\frac{d}{dx}\left[\frac{1}{F(x)}\right] = \frac{-1}{(F(x))^2} F'(x)$

$$= (-1) \frac{d^2}{ds^2}\left[\frac{-1}{(s+3)^2}\right]$$

$$= (-1) \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{-1}{(s+3)^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{(s+3)^2}\right)\right]$$

use $\frac{d}{dx}\left[\frac{1}{(F(x))^n}\right] = \frac{-n}{(F(x))^{n+1}} \times F'(x)$

$$= \frac{d}{ds}\left(\frac{-2}{(s+3)^3}\right)$$

$$= -2 \frac{d}{ds}\left(\frac{1}{(s+3)^3}\right)$$

$T \to a \to e^x$

$= -2 \times \frac{-3}{(s+3)^4}$

$\therefore \boxed{L[t^3 e^{-3t}] = \frac{6}{(s+3)^4}}$

7) $L[t^2 e^{-at} \sinh at]$

$\to$

Let $F(t) = \sinh at$

$L[F(t)] = L[\sinh at]$

$= \frac{a}{s^2-a^2}$

$= \bar{F_1}(s)$

$L[t^2 F(t)] = (-1)^2 \frac{d^2}{ds^2}[\bar{F_1}(s)]$

$= 1 \frac{d^2}{ds^2}\left[\frac{a}{s^2-a^2}\right]$

$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)\right]$

$= a \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s^2-a^2}\right)\right]$

Use $\frac{d}{dx}\left(\frac{1}{F(x)}\right) = \frac{1}{[F(x)]^2} F'(x)$

$= a \frac{d}{ds}\left[\frac{-1}{(s^2-a^2)^2} \times 2s\right]$

$$= -2a \frac{d}{ds}\left[\frac{s}{(s^2-a^2)^2}\right]$$

Use $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{DN' - ND'}{D^2}$

$$= -2a\left[\frac{(s^2-a^2)^2 \times \frac{d}{ds}(s) - s\frac{d}{ds}(s^2-a^2)^2}{[(s^2-a^2)^2]^2}\right]$$

$$= -2a\left[\frac{(s^2-a^2)^2 \times 1 - s[2(s^2-a^2)\times(2s)]}{[(s^2-a^2)^2]^2}\right]$$

$$= -2a\left[\frac{(s^2-a^2)^2 - s[4s(s^2-a^2)]}{(s^2-a^2)^4}\right]$$

$$= -2a\left[\frac{(s^2-a^2)^2 - 4s^2(s^2-a^2)}{(s^2-a^2)^4}\right]$$

$$= -2a(s^2-a^2)\left[\frac{s^2-a^2-4s^2}{(s^2-a^2)^{\cancel{4}}}\right]$$

$$= -2a\left[\frac{-a^2-3s^2}{(s^2-a^2)^3}\right]$$

$$= \frac{2a(3s^2+a^2)}{(s^2-a^2)^3}$$

$$= \overline{F}_2(s)$$

$$L[e^{-at}F(t)] = \overline{F}_2(s+a)$$

$$L[e^{-at}t^2 \sinh at] = \overline{F}_2(s+a)$$

$$\boxed{L[t^2 e^{-at} \cosh at] = \frac{2a[3(s+a)^2 + a^2]}{[(s+a)^2 - a^2]^3}}$$

Laplace Transform of Integral.

If $L[F(t)] = \bar{F}(s)$

then $L\left[\int_0^t f(t)\,dt\right] = \frac{\bar{F}(s)}{s}$

1. $L\left[\int_0^t \sin 2t \, dt\right]$

Let $F(t) = \sin 2t$

$L[F(t)] = L[\sin 2t]$

$= \frac{2}{s^2+4}$

$\therefore = \bar{F}(s)$

$\therefore L\left[\int_0^t f(t)\,dt\right] = \frac{\bar{F}(s)}{s}$

$$\boxed{L\left[\int_0^t \sin 2t\,dt\right] = \frac{2}{s(s^2+4)}}$$

H.W

1) $L[t \cos 3t \cos 4t]$

2) $L[t e^t \sin^2 2t]$

1) $L[t\cos 3t \cdot \cos 4t]$

$\rightarrow$ Let $F(t) = \cos 3t \cdot \cos 4t$

$2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

$2\cos 3t \cdot \cos 4t = \cos(3t+4t) + \cos(3t-4t)$

$= \cos(7t) + \cos(-t)$

$= \cos 7t + \cos t$

$\cos 3t \cos 4t = \frac{1}{2}[\cos 7t + \cos t]$

$L[\cos 3t \cdot \cos 4t] = \frac{1}{2} L[\cos 7t + \cos t]$

$$= \frac{1}{2}\left[\frac{s}{s^2+49} + \frac{s}{s^2+1}\right]$$

$= \bar{F}(s)$

$$L[t\cos 3t \cdot \cos 4t] = \frac{1}{2}(-1)\frac{d}{ds}\left[\frac{s}{s^2+49} + \frac{s}{s^2+1}\right]$$

$$= \frac{-1}{2}\left[\frac{d}{ds}\left(\frac{s}{s^2+49}\right) + \frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$$

use $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{-1}{2}\left[\left(\frac{(s^2+49)\times 1 - s(2s+0)}{(s^2+49)^2}\right) + \left(\frac{(s^2+1)\times 1 - s(2s+0)}{(s^2+1)^2}\right)\right]$$

$$= \frac{-1}{2}\left[\frac{s^2+49-2s^2}{(s^2+49)^2} + \frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{-s^2+49}{(s^2+49)^2} + \frac{-s^2+1}{(s^2+1)^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-49}{(s^2+49)^2} + \frac{s^2-1}{(s^2+1)^2}\right]$$

$$\therefore \boxed{L[t \cdot \cos 3t \cdot \cos 4t] = \frac{1}{2}\left[\frac{s^2-49}{(s^2+49)^2} + \frac{s^2-1}{(s^2+1)^2}\right]}$$

2) $L[te^t \sin^2 2t]$

$\rightarrow$ let $f(t) = \sin^2 2t$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\therefore \sin^2 2t = \frac{1-\cos 4t}{2}$$

$$\sin^2 2t = \frac{1}{2}[1-\cos 4t]$$

$$L[\sin^2 2t] = \frac{1}{2} L[1-\cos 4t]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$= \bar{F}_1(s)$$

$$L[t \cdot \sin^2 2t] = (-1)\frac{d}{ds}(\bar{F}_1(s))$$

$$= -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)\right]$$

$$= \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$= \frac{-1}{2}\left[\frac{d}{ds}\left(\frac{1}{s}\right) - \frac{d}{ds}\left(\frac{s}{s^2+16}\right)\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} - \left[\frac{(s^2+16)\times 1 - s(2s+0)}{(s^2+16)^2}\right]\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} - \left(\frac{s^2+16-2s^2}{(s^2+16)^2}\right)\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} - \left(\frac{-s^2+16}{(s^2+16)^2}\right)\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{s^2-16}{(s^2+16)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} - \frac{s^2-16}{(s^2+16)^2}\right]$$

$$= \bar{F_2}(s)$$

Now,

$$L[e^{at} F(t)] = \bar{F}(s-a)$$

$$\therefore L[te^t \sin^2 2t] = \bar{F_2}(s-1)$$

$$\therefore L[te^t \sin^2 2t] = \frac{1}{2}\left[\frac{1}{(s-1)^2} - \frac{(s-1)^2-16}{((s-1)^2+16)^2}\right]$$

$$\boxed{L[te^t \sin^2 2t] = \frac{1}{2}\left[\frac{1}{(s-1)^2} - \frac{(s-1)^2-16}{[(s-1)^2+16]^2}\right]}$$

Laplace Transform of Integral.

$L[F(t)] = \bar{F}(s)$

$L\left[\int_0^t F(t)\,dt\right] = \frac{\bar{F}(s)}{s}$

1) $L\left[\int_0^t e^{-t} \cos t\,dt\right]$

Let $F(t) = \cos t$

$L[F(t)] = L[\cos t]$

$L[\cos t] = \frac{s}{s^2+1}$

$= \bar{F}_1(s)$

$L[e^{-at} \cancel{\cos} F(t)] = \bar{F}_1(s+a)$

$L[e^{-t} \cos t] = \bar{F}_1(s+1)$

$= \frac{s+1}{(s+1)^2+1}$

$= \bar{F}_2(s)$

$L\left[\int_0^t F(t)\,dt\right] = \frac{\bar{F}_2(s)}{s}$

$L\left[\int_0^t e^{-t} \cos t\,dt\right] = \frac{s+1}{(s+1)^2+1} \times \frac{1}{s}$

$$\boxed{L\left[\int_0^t e^{-t} \cos t\,dt\right] = \frac{s+1}{s[(s+1)^2+1]}}$$

2) $L\left[\int_0^t e^{t} \frac{\sin t}{t}\,dt\right]$

let $F(t) = \sin t$

$$L[\sin t] = \frac{1}{s^2+1} = \bar{F_1}(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F_1}(s)\, ds$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\, ds$$

$t\{$ Use $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$L\left[\frac{\sin t}{t}\right] = \frac{1}{1}\left[\tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \pi/2 - \tan^{-1}(s)$$

$$= \cot^{-1}(s) = \bar{F_2}(s)$$

$$L[e^{at} F_2(t)] = \bar{F_2}(s-a)$$

$$L\left[e^t \frac{\sin t}{t}\right] = \cot^{-1}(s-1) = \bar{F_3}(s)$$

$$L\left[\int_0^t F_3(t)\, dt\right] = \frac{\bar{F_3}(s)}{s}$$

$$\boxed{L\left[\int_0^t e^t \frac{\sin t}{t}\, dt\right] = \frac{\cot^{-1}(s-1)}{s}}$$

3. $L\left[\int_0^t \left(\frac{e^{-at}-e^{-bt}}{t}\right) dt\right]$

→ Let $F(t) = e^{-at} - e^{-bt}$

$$L[e^{-at} - e^{-bt}] = \frac{1}{s+a} - \frac{1}{s-b} = \bar{F_1}(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}(s)\, ds$$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s-b}\right) ds$$

use $\int \frac{F'(x)}{F(x)} dx = \log F(x)$

$$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \Big[\log(s+a) - \log(s+b)\Big]_s^\infty$$

$$= \Big[0 - 0 - (\log(s+a) - \log(s+b))\Big]$$

$$= \log(s+b) - \log(s+a)$$

use $\log a - \log b = \log(a/b)$

$$= \log\left(\frac{s+b}{s+a}\right) = \bar{F_2}(s)$$

$$L\left[\int_0^t F(t)\, dt\right] = \frac{\bar{F_2}(s)}{s}$$

$$\boxed{L\left[\int_0^t \left(\frac{e^{-at}-e^{-bt}}{t}\right) dt\right] = \frac{1}{s} \times \log\left(\frac{s+b}{s+a}\right)}$$

4. $L\left[\int_0^t \left(\frac{\cos at - \cos bt}{t}\right) dt\right]$

→ Let $F(t) = \cos at - \cos bt$

$L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = \bar{F}_1(s)$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}_1(s)\, ds$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

use $\int \frac{F'(x)}{F(x)}\, dx = \log F(x)$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - 0 - \left(\log(s^2+a^2) - \log(s^2+b^2)\right)\right]$$

$$= \frac{1}{2}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

use $a \log b = \log b^a$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

trig → algebric → integral → expo

DATE

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}} = \bar{F}_2(s)$$

$$L\left[\int_0^t F(t)\,dt\right] = \frac{\bar{F}_2(s)}{s}$$

$$\therefore \boxed{L\left[\int_0^t \left(\frac{\cos at - \cos bt}{t}\right) dt\right] = \frac{1}{s}\log\sqrt{\frac{s^2+b^2}{s^2+a^2}}}$$

*** 5. $L\left[e^{-4t}\int_0^t \frac{\sin 3t}{t}\,dt\right]$

→ Let $F(t) = \sin 3t$

$$L[\sin 3t] = \frac{3}{s^2+9} = \bar{F}_1(s)$$

$$L\left[\frac{F(t)}{t}\right] = \int_s^\infty \bar{F}_1(s)\,ds$$

$$= \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

Use $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$$= 3\int_s^\infty \frac{1}{s^2+3^2}\,ds$$

$$= 3 \times \frac{1}{3}\left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)$$

$$= \cot^{-1}(s/3)$$

$$= \overline{F_2}(s)$$

$$L\left[\int_0^t \cancel{s} F(t)\, dt\right] = \frac{\overline{F_2}(s)}{s}$$

$$L\left[\int_0^t \frac{\sin 3t}{t}\, dt\right] = \frac{1}{s}\cot^{-1}\left(\frac{s}{3}\right)$$

$$= \overline{F_3}(s)$$

$$L[e^{-at} F(t) \cancel{dt}] = \overline{F_3}(s+a)$$

$$\boxed{L\left[e^{-4t}\int_0^t \frac{\sin 3t}{t}\, dt\right] = \frac{1}{s+4}\cot^{-1}\left(\frac{s+4}{3}\right)}$$

6. $L\left[\int_0^t t e^{-t} \sin 4t\, dt\right]$

→ Let $F(t) = \sin 4t$

$$L[\sin 4t] = \frac{4}{s^2+16} = \overline{F_1}(s)$$

$$L[t F(t)] = (-1) \frac{d}{ds}(\overline{F_1}(s))$$

$$L[t \sin 4t] = (-1)^1 \frac{d}{ds}\left[\frac{4}{s^2+16}\right]$$

$$= (-1) \times 4 \frac{d}{ds}\left(\frac{1}{s^2+16}\right)$$

$$\frac{d}{dx}\left(\frac{1}{F(x)}\right) = \frac{-1}{[F(x)]^2} \times F'(x)$$

$$= -4 \times \frac{-1}{(s^2+16)^2} \times 2s$$

$$= \frac{8s}{(s^2+16)^2}$$

$$= \bar{F}_2(s)$$

$$L[e^{-at} F(t)] = \bar{F}_2(s+a)$$

$$L[e^{-t} t \sin 4t] = \frac{8(s+1)}{[(s+1)^2+16]^2} = \bar{F}_3(s)$$

$$L\left[\int_0^t F(t)\,dt\right] = \frac{\bar{F}_3(s)}{s}$$

$$\boxed{L\left[\int_0^t te^{-t} \sin 4t\,dt\right] = \frac{8(s+1)}{s[(s+1)^2+16]^2}}$$

7. $L\left[\int_0^t \int_0^t \int_0^t \cos at\,dt\,dt\,dt\right]$

→ Let $F(t) = \cos at$

$$L[\cos at] = \frac{s}{s^2+a^2} = \bar{F}_1(s)$$

$$L\left[\int_0^t F(t)\,dt\right] = \frac{\bar{F}_1(s)}{s}$$

$$\therefore L\left[\cancel{\int_0^t}\int_0^t F(t)\,dt\right] = \frac{s}{s(s^2+a^2)}$$

$$L\left[\cancel{\int_0^t}\int_0^t \cos at\,dt\right] = \frac{1}{(s^2+a^2)} = \bar{F}_2(s)$$

$$L\left[\int_0^t F(t)\,dt\right] = \frac{\bar{F}_2(s)}{s}$$

$$L\left[\int_0^t\int_0^t\int_0^t \cos at\,dt\right] = \frac{1}{s(s^2+a^2)} = \bar{F}_3(s)$$

$$L\left[\int_0^t F(t)\,dt\right] = \frac{\bar{F}_3(s)}{s}$$

$$\boxed{L\left[\int_0^t\int_0^t\int_0^t \cos at\,dt\,dt\,dt\right] = \frac{1}{s^2(s^2+a^2)}}$$

Error Function :-

The error function is denoted as $erf(\sqrt{t})$. It is defined as,

$$\boxed{erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx}$$

$$erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$$

put $x^2 = t$

diff. w.r.t. 't'

$$2x\frac{dx}{dt} = 1$$

$$2x\,dx = dt$$

$$dx = \frac{1}{2x}\,dt$$

$$x^2 = t$$

$$\therefore x = t^{1/2}$$

$$\Rightarrow dx = \frac{1}{2t^{1/2}} dt$$

$$dx = \frac{1}{2} t^{-1/2} dt$$

as $x = 0$ $\qquad\qquad t = 0$

$x^2 = t$

$0 = t$

$\Rightarrow t = 0$

$x = \sqrt{t}$ $\qquad\qquad t = t$

$x^2 = t$

$(\sqrt{t})^2 = t$

$\therefore t = t$

$$\therefore erF(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx$$

$$erF(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t} \frac{t^{-1/2}}{2} dt$$

Taking L.T on both sides

$$L[erF(\sqrt{t})] = \frac{2}{\sqrt{\pi}} L\left[\int_0^t \frac{e^{-t} t^{-1/2}}{2} dt\right]$$

$$= \frac{2}{\sqrt{\pi}} \times \frac{1}{2} L\left[\int_0^t e^{-t} t^{-1/2} dt\right] \quad \text{— (A)}$$

Consider,

$$= L\left[\int_0^t e^{-t}\, t^{-1/2}\, dt\right]$$

let $F(t) = t^{-1/2}$

$$L[t^{-1/2}] = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}}$$

$$= \frac{\Gamma(1/2)}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \bar{F_1}(s)$$

$$L[e^{-at} F(t)] = \bar{F_1}(s+a)$$

$$L[e^{-t}\, t^{-1/2}] = \frac{\sqrt{\pi}}{\sqrt{s+1}} = \bar{F_2}(s)$$

$$L\left[\int_0^t F(t)\, dt\right] = \frac{\bar{F_2}(s)}{s}$$

$$L\left[\int_0^t e^{-t}\, t^{-1/2}\, dt\right] = \frac{\sqrt{\pi}}{s\sqrt{s+1}} \longrightarrow (B)$$

From eq^n A & B

$$L[\operatorname{erf}(\sqrt{t})] = \frac{1}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\boxed{L[\operatorname{erf}(\sqrt{t})] = \frac{1}{s\sqrt{s+1}}}$$

** Heaviside Unit Step Function.

- $u(t-a) = 0 \quad t<a$
  $= 1 \quad t>a$

- $u(t) = 0 \quad t<0$
  $= 1 \quad t>0$

- $F(t)\,u(t-a) = 0 \quad t<a$
  $= F(t) \quad t>a$

- $L[u(t-a)] = \frac{e^{-as}}{s}$

- $L[u(t)] = \frac{1}{s}$

Imp - $L[F(t-a)\,u(t-a)] = e^{-as}\,L[F(t)]$

Examples

1. Find the laplace transform of

$$F(t) = \begin{cases} 2 & 0<t<\pi \\ 0 & \pi<t<2\pi \\ \sin t & t>2\pi \end{cases}$$

→

$$F(t) = 2[u(t-0) - u(t-\pi)] + 0[u(t-\pi) - u(t-2\pi)] + \sin t\,[u(t-2\pi)]$$

$$= 2u(t) - 2u(t-\pi) + 0 + \sin t \cdot u(t-2\pi)$$

$$= 2u(t) - 2u(t-\pi) + \sin t \cdot u(t-2\pi)$$

$$= 2u(t) - 2u(t-\pi) + \sin [(t-2\pi)+2\pi]\, u(t-2\pi)$$

use $\sin(\theta + 2\pi) = \sin\theta$

$$= 2u(t) - 2u(t-\pi) + \sin(t-2\pi) \cdot u(t-2\pi)$$

Taking L.T on both sides

$$L[F(t)] = 2L[u(t)] - 2L[u(t-\pi)] + L[\sin(t-2\pi) \cdot u(t-2\pi)]$$

$$= \frac{2}{s} - \frac{2e^{-\pi s}}{s} + e^{-2\pi s} L[\sin t]$$

$$\boxed{L[F(t)] = \frac{2}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}}$$

2. Find L.T of

$$F(t) = \begin{cases} 1 & 0<t<2 \\ 2 & 2<t<4 \\ 3 & 4<t<6 \\ 0 & t>6 \end{cases}$$

→

$$F(t) = 1[u(t-0) - u(t-2)] + 2[u(t-2) - u(t-4)] + 3[u(t-4) - u(t-6)] + 0[u(t-6)]$$

$$= u(t) - u(t-2) + 2u(t-2) - 2u(t-4) + 3u(t-4) - 3u(t-6) + 0$$

$$= u(t) - u(t-2) + 2u(t-2) - 2u(t-4) + 3u(t-4) - 3u(t-6)$$

$$= u(t) + u(t-2) + u(t-4) - 3u(t-6)$$

Taking L.T on both sides.

$$L[F(t)] = L[u(t)] + L[u(t-2)] + L[u(t-4)] - 3L[u(t-6)]$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s} + \frac{e^{-4s}}{s} - 3\frac{e^{-6s}}{s}$$

$$\boxed{L[F(t)] = \frac{1}{s}\left[1 + e^{-2s} + e^{-4s} - 3e^{-6s}\right]}$$

3. Find L.T of $F(t) = 0 \qquad 0 < t < \pi/2$

$= \sin t \qquad t > \pi/2$

$$F(t) = 0\,[u(t-0) - u(t-\pi/2)] + \sin t\,[u(t-\pi/2)]$$

$$= \sin t \cdot u(t-\pi/2)$$

$$= \sin\left(t - \frac{\pi}{2} + \frac{\pi}{2}\right) u(t-\pi/2)$$

$$= \sin\left[\underline{(t-\pi/2)} + \pi/2\right] u(t-\pi/2)$$

use $\sin(\theta + \pi/2) = \cos\theta$

$$= \overset{\cos}{\cancel{\sin}}\,(t-\pi/2) \cdot u(t-\pi/2)$$

$$L[F(t)] = L[\cos(t-\pi/2) \cdot u(t-\pi/2)]$$

Use $L[F(t-a)\,u(t-a)] = e^{-as} L[F(t)]$

$$= e^{-\frac{\pi}{2}s} L[\cos t]$$

$$\boxed{L[F(t)] = e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2+1}}$$

4. Find $L[e^{-4t}u(t-1)]$

→

Consider,

$$e^{-4t}u(t-1) = e^{-4(t-1+1)}u(t-1)$$
$$= e^{-4(t-1)-4}u(t-1)$$
$$= e^{-4(t-1)}e^{-4}u(t-1)$$
$$e^{-4t}u(t-1) = e^{-4}[e^{-4(t-1)}u(t-1)]$$

Taking L.T on both sides

$$L[e^{-4t}u(t-1)] = e^{-4}L[e^{-4(t-1)}u(t-1)]$$

use $L[F(t-a)u(t-a)] = e^{-as}\bar{F}(s) = e^{-as}L[F(t)]$

$$= e^{-4}e^{-s}L[e^{-4t}]$$
$$= e^{-s-4}\frac{1}{s+4}$$

$$\boxed{L[e^{-4t}u(t-1)] = \frac{e^{-(s+4)}}{s+4}}$$

H.W 5

$$F(t) = t-1 \qquad 1<t<2$$
$$= 3-t \qquad 2<t<3$$

New type.

1. Solve $\int_0^\infty e^{-2t}\,t\sin 3t\,dt$

→ Let $F(t) = \sin 3t$

$$L[\sin 3t] = \frac{3}{s^2+9} = \bar{F}_1(s)$$

$$L[t\sin 3t) = \frac{-d}{ds}[\bar{F}_1(s)]$$

$$= \frac{-d}{ds}\left[\frac{3}{s^2+9}\right]$$

$$= -3\left[\frac{-1}{(s^2+9)^2}(2s+0)\right]$$

$$= \frac{6s}{(s^2+9)^2} = \bar{F}_2(s)$$

$$\int_0^\infty e^{-2t}\, t \sin 3t \, dt = \bar{F}_2(2)$$

$$= \frac{6 \times 2}{(2^2+9)^2}$$

$$\boxed{\int_0^\infty e^{-2t}\, t \sin 3t \, dt = \frac{12}{169}}$$

2. Solve $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} \, dt$

→ Let $F(t) = \sin^2 t$

$$L[\sin^2 t] = \frac{1}{2} L[1 - \cos 2t]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] = \bar{F}_1(s)$$

$$L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty \bar{F}_1(s)\, ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \frac{1}{2} \times 0 - \left(\log s - \frac{1}{2}\log(s^2+4)\right)\right]$$

$$= \frac{1}{2}\left[-\log s + \frac{1}{2}\log(s^2+4)\right]$$

$$a \log b = \log b^a$$

$$= \frac{1}{2}\left[\log(s^2+4)^{1/2} - \log s\right]$$

$$\log a - \log b = \log(a/b)$$

$$= \frac{1}{2}\left[\log \frac{(s^2+4)^{1/2}}{s}\right]$$

$$= \frac{1}{2}\log \frac{\sqrt{s^2+4}}{s} = \bar{F}_2(s)$$

$$\int_s^\infty e^{-t} \frac{\sin^2 t}{t}\, dt = \bar{F}_2(1) = \frac{1}{2}\log \frac{\sqrt{1+4}}{1}$$

$$\boxed{\int_s^\infty e^{-t} \frac{\sin^2 t}{t}\, dt = \frac{1}{2}\log\sqrt{5}}$$

3. Solve $\int_0^\infty \left(\frac{e^{-3t} - e^{-6t}}{t}\right) dt$

M-I

$\Rightarrow \int_s^\infty \left(\frac{e^{-3t} - e^{-3t} \cdot e^{-3t}}{t}\right) dt$

$$= \int_0^\infty \frac{e^{-3t}(1-e^{-3t})}{t}\, dt$$

Let $F(t) = 1-e^{-3t}$

$$L[1-e^{-3t}] = \frac{1}{s} - \frac{1}{s+3}$$

$$L\left[\frac{1-e^{-3t}}{t}\right] = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+3}\right) ds$$

$$= \Big[\log s - \log(s+3)\Big]_s^\infty$$

$$= \Big[0 - 0 - [\log s - \log(s+3)]\Big]$$

$$= -\log s + \log(s+3)$$

$$= \log \frac{s+3}{s}$$

$$= \bar{F}_2(s)$$

$$\int_0^\infty \frac{e^{-3t}(1-e^{-3t})}{t} = \bar{F}_2(3)$$

$$= \log \frac{3+3}{3}$$

$$\boxed{\int_0^\infty \frac{e^{-3t}(1-e^{-3t})}{t} = \log 2}$$

M-II Consider $\int_0^\infty e^{0} \left(\frac{e^{-3t} - e^{-6t}}{t}\right) dt$

DATE

H.W. 4. $\int_0^\infty e^{-3t}\, t \cos t \, dt = \frac{2}{25}$

$\rightarrow$ $\int_0^\infty e^{-3t} (t \cos t \, dt$.

Let $F(t) = \cos t$

$L[\cos t] = \frac{s}{s^2+1}$

$L\left[\frac{\cos t}{t}\right] = \int_s^\infty \frac{s}{s^2+1} \, ds$

$= \frac{1}{2} \int_s^\infty \frac{2s}{s^2+1} \, ds$

$= \frac{1}{2} \left[\log(s^2+1)\right]_s^\infty$

$= \frac{1}{2} \left[0 - \log(s^2+1)\right]$

$= -\frac{1}{2} \log(s^2+1)$

$=$

H.W 4. $\int_0^\infty e^{-3t}\, t \cos t \, dt = \frac{2}{25}$

$\rightarrow$ Let $F(t) = \cos t \, dt$

$L[\cos t] = \frac{s}{s^2+1} = \bar{F_1}(s)$

$L[t \cos t] = -\frac{d}{ds} \left(\frac{s}{s^2+1}\right)$

$$= -\left[\frac{(s^2+1)\times 1 - s(2s)}{(s^2+1)^2}\right]$$

$$= \frac{-(s^2+1-2s^2)}{(s^2+1)^2}$$

$$L[t\cos t] = \frac{-1+s^2}{(s^2+1)^2}$$

$$= \bar{F}_2(s)$$

$$\int_0^\infty e^{-3t}\, t\cos t\, dt = \bar{F}_2(3)$$

$$\int_0^\infty e^{-3t}\, t\cos t\, dt = \frac{-1+3^2}{(3^2+1)^2}$$

$$= \frac{-1+9}{(9+1)^2}$$

$$= \frac{8}{(10)^2}$$

$$= \frac{+8}{100}$$

$$\boxed{\int_0^\infty e^{-3t}\, t\cos t\, dt = +\frac{2}{25}}$$

H.W

$$F(t) = t-1 \quad 1<t<2$$
$$= 3-t \quad 2<t<3$$

$\longrightarrow$ $$F(t) = t-1 \quad 1<t<2$$
$$= -(t-3) \quad 2<t<3$$

$$F(t) = (t-1)[u(t-1)-u(t-2)] - (t-3)[u(t-2)-u(t-3)]$$

$$= (t-1)[u(t-1)-u(t-2)] - (t-3)[u(t-2)-u(t-3)]$$

$$= (t-1)u(t-1) - (t-1)u(t-2) - (t-3)u(t-2) + (t-3)u(t-3)$$

$$= (t-1)u(t-1) + u(t-2)(-t+1-t+3) + (t-3)u(t-3)$$

$$= (t-1)u(t-1) + u(t-2)(-2t+4) + (t-3)u(t-3)$$

$$= (t-1)u(t-1) - 2u(t-2)(t-2) + (t-3)u(t-3)$$

$$\cancel{= (t-1)u(t-1) - 2u(t-2)^2 + (t-3)u(t-3)}$$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

$$= e^{-s}\left(\frac{1}{s^2}\right) - 2e^{-2s}\left(\frac{1}{s^2}\right) + e^{-3s}\left(\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2}\left[e^{-s} - 2e^{-2s} + e^{-3s}\right]$$

$$\boxed{F(t) = \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2}}$$

* Laplace Transform of Periodic Function.

$$L[F(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt$$

where T is period of function

- $\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$

(LIATE)

- $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

* Examples.

1) Find L.T of $F(t) = \sin\left(\frac{\pi t}{a}\right)$ $0<t<T$ and period of function $T=a$

→ $$L[F(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\, dt$$

$T=a$

$$L[F(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} \sin\left(\frac{\pi}{a}t\right) dt$$

use $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

$a=-s$, $b=\frac{\pi}{a}$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{-st}}{(-s)^2+\left(\frac{\pi}{a}\right)^2}\left(-s\sin\left(\frac{\pi}{a}t\right)-\frac{\pi}{a}\cos\left(\frac{\pi}{a}t\right)\right)\right]_0^a$$

$$= \frac{1}{1-e^{-as}}\left\{\left[\frac{e^{-as}}{s^2+\frac{\pi^2}{a^2}}\left(-s\sin\left(\frac{\pi}{a}\times a\right)-\frac{\pi}{a}\cos\left(\frac{\pi}{a}\times a\right)\right)\right]\right.$$

$$\left.-\left[\frac{e^0}{s^2+\frac{\pi^2}{a^2}}\left(-s\sin 0-\frac{\pi}{a}\cos 0\right)\right]\right\}$$

$$= \frac{1}{1-e^{-as}}\left\{\left[\frac{e^{-as}}{\frac{a^2s^2+\pi^2}{a^2}}\left(0+\frac{\pi}{a}\right)\right]-\left[\frac{1}{\frac{a^2s^2+\pi^2}{a^2}}\left(0-\frac{\pi}{a}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-as}}\left[\frac{a^2e^{-as}}{a^2s^2+\pi^2}\times\frac{\pi}{a}+\frac{a^2}{a^2s^2+\pi^2}\times\frac{\pi}{a}\right]$$

$$= \frac{1}{1-e^{-as}}\times\frac{a\pi}{a^2s^2+\pi^2}\left[e^{-as}+1\right]$$

$$\boxed{L[F(t)] = \left[\frac{1+e^{-as}}{1-e^{-as}}\right]\frac{a\pi}{a^2s^2+\pi^2}}$$

2. Find L.T for the periodic function.

$F(t) = t \qquad 0<t<c$

$\quad = 2c-t \qquad c<t<2c$

→

$$L[F(t)] = \frac{1}{1-e^{-st}}\int_0^T e^{-st}F(t)\,dt$$

$T = 2c$

$$L[F(t) = \frac{1}{1-e^{-2cs}}\int_0^{2c} e^{-st}F(t)\,dt$$

$$= \frac{1}{1-e^{-2cs}}\left[\underbrace{\int_0^c e^{-st}\, t\, dt}_{A_1} + \underbrace{\int_c^{2c} (2c-t)\, e^{-st}\, dt}_{A_2}\right] \quad \text{— (A)}$$

Consider,

$$A_1 = \int_0^c t e^{-st}\, dt$$

$$= \left[t\frac{e^{-st}}{-s} - 1 \times \frac{1}{(-s)} \frac{e^{-st}}{(-s)} \right]_0^c$$

$$= \left[\left(-\frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$A_1 = \frac{1}{s^2} - \frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2}$$

Consider.

$$A_2 = \int_c^{2c} (2c-t)\, e^{-st}\, dt$$

$$= \left[(2c-t)\frac{e^{-st}}{-s} - (-1)\left(\frac{1}{-s}\right)\frac{e^{-st}}{(-s)} \right]_c^{2c}$$

$$= \left[0 + \frac{e^{-2cs}}{s^2} - \left(-\frac{(2c-c)}{s} e^{-cs} + \frac{1}{s^2} e^{-cs} \right) \right]$$

$$= \left[\frac{e^{-2cs}}{s^2} - \left(-\frac{ce^{-cs}}{s} + \frac{e^{-cs}}{s^2} \right) \right]$$

$$A_2 = \frac{e^{-2cs}}{s^2} + \frac{ce^{-cs}}{s} - \frac{e^{-cs}}{s^2}$$

Put A_1 & A_2 in A.

$$L[F(t)] = \frac{1}{1-e^{-2cs}}\left[\frac{1}{s^2} - \cancel{\frac{ce^{-cs}}{s}} - \frac{e^{-cs}}{s^2} + \frac{e^{-2cs}}{s^2} + \cancel{\frac{ce^{-cs}}{s}} - \frac{e^{-cs}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2cs}}\left[\frac{1}{s^2} - \frac{e^{-cs}}{s^2} - \frac{e^{-cs}}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2cs})}\left[1 - 2e^{-cs}\right]$$

$$\therefore \boxed{L[F(t)] = \frac{(1-2e^{-cs})}{s^2(1-e^{-2cs})}}$$